AN INVESTIGATION OF STRATEGIES USED TO SOLVE A SIMPLE DEDUCTIVE EXERCISE IN GEOMETRY

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Abstract
The purpose of this paper is to report the analysis of senior secondary students' proof using deduction on an exercise based on a parallelogram. The question was designed so that it could be solved by recourse to a relatively little used definition of a parallelogram. The results showed that the better students did make use of the definition. However, the majority of students who chose to use a deductive approach, relied initially on congruency concepts. For these responses a clear difference in quality was identified. Because of this work a general framework of growth, based on the SOLO Taxonomy, is suggested. Following from this a possible structure of the nature of early Level 4 (deductive) thinking identified by the van Hiele theory is hypothesised.

Background

The van Hiele Theory (van Hiele, 1986) offers a conceptual framework from which to view student thinking processes in Geometry. Despite criticism of the theory, numerous research studies have offered broad support for the framework, see for example Clements & Battista (1992). In particular, the notion of a series of levels has had considerable empirical support. The focus of this paper is primarily concerned with one of these levels, namely, Level 4. This level can be generally identified in Geometry by the following characteristics: proofs can be developed; definitions in terms of minimum properties can be produced, i.e., necessary and sufficient conditions are understood; and, the role of deduction is understood.

It is relevant that this description of Level 4 thinking is consistent with the types of responses that would be coded within the Formal mode of the SOLO Taxonomy of Biggs and Collis (Biggs & Collis, 1991). Previous reports by one of the writers (see for example Pegg (1992a) and Pegg & Faithfull (1993)) have shown that the SOLO Taxonomy with its modes of functioning and levels of attainment within modes is a valuable tool in developing further the meanings of the Levels proposed by the van Hiele model. For example the existence of cycles of levels under the general headings of:

unistructural	-	focussing on one aspect
multistructural	-	focussing on several independent aspects
relational	-	having an overview of several aspects

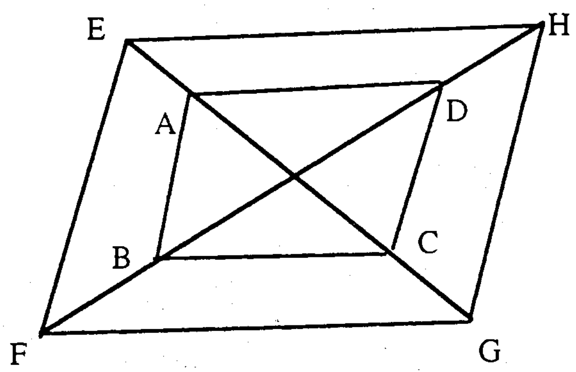
within the Formal mode offer the potential to elaborate more fully the nature of deductive thinking.

Design

This paper represents the results of an analysis of a question which was taken from a larger set of questions given to 55 senior secondary students (16 - 18 year olds). The students in the study had undertaken a relatively extensive geometry programme in the junior secondary school and all were currently undertaking a deductive geometry course as part of their senior secondary mathematics programme. The students were chosen to be representative of the type of abilities present in the top 60 % of the senior secondary school cohort.

The purpose of the main investigation was to explore students' understanding of properties and geometrical relationships associated with parallelograms and compare how this knowledge translated into solution processes in a series of deductive exercises. These exercises were designed specifically so they can be solved by recourse to various strategies, in particular congruency and definitions.

The question analysed in this paper was one of six questions within the same general area. The proof can be addressed by either resorting to the use of congruent triangles and/or necessary and sufficient conditions. It is the consideration of these two options that is the main focus of this paper. The question asked is given below and is representative of the other questions in the study.



ABCD is a parallelogram.
AE = CG BF =DH
Prove EH = FG

The question was asked within an interview situation. Students were provided with paper for working and asked to speak aloud while solving the question. Their comments were taped and later transcribed. The students' work was collected and analysed along with the transcript of the interview. The interviewer's role was: to ensure students understood the question; to seek clarification when the student's answers were vague; and, to answer questions the students might ask. However, the interviewer did not provide help other than that requested by the student. The net result of this process may have been to allow some students to answer beyond their normal functional level.

Research Questions

There were three questions that guided the research.

1. What strategies do students employ whilst solving the deductive exercise posed?
2. Is there an identifiable hierarchy in the quality of the responses provided?
3. What implication can be drawn from the results when they are considered within the framework offered by the SOLO Taxonomy and how does this help in clarifying apects of van Hiele's Level 4?

Analysis

Several groupings of responses were identified. Thirty-eight students (69%) took a deductive stance in their response and, as such, would represent Level 4 thinking (van Hiele Theory) and be coded within the Formal mode in the SOLO Taxonomy. The remaining seventeen students did not employ a deductive argument but instead chose to work with the diagram by applying a form of natural logic: characteristics common to Level 3 and 2 thinking (van Hiele Theory) and concrete symbolic responses in the SOLO Taxonomy. Below is a more detailed analysis of the main categories of responses with typical examples taken from the transcript of interview to help illustrate their nature.

Group A

> Student: *"If ABCD is a parallelogram then the diagonals should bisect each other in the middle, so if the same length is added to the diagonals in the same line then diagonals remain bisected and then the line drawn to join the lines from the diagonals will be parallel and equal so therefore EH will equal FG."*

The student has a clear overview of the question. The response is concise, precise and demonstrates that the student is 'comfortable' with the sufficiency of the property that the diagonals of a parallelogram bisect each other. Also impressive is the way the student sees the solution in terms of a general principle.

Group B

> Student: *"Well, diagonals of original parallelogram are projected out a little and projected out equal amounts, means all properties of diagonals for the one we're given is a parallelogram remain for — I'm not trying to prove it's a parallelogram am I?"*
>
> Interviewer: *"You've got to prove EH = FG."*
>
> Student: *"Oh we don't need that then. Same idea, diagonals of a parallelogram are projected out - which means that angle F, Oh! (there is) nothing in the middle".*
>
> *Interviewer: "Call it O or X, give it a label."*

Student: *"Angle FOG equals angle EOH because it's given, so one angle equal also given that FO equals OG because diagonals of parallelogram are same, bisect each other, extended same amount so they are equal so because got two sides and included angle it means triangles are equal and EH = FG."*

Responses here were not quite as confident as in group A, although the sufficiency of the diagonals being bisected appears to be accepted. In this case, the student needed to continue her initial thoughts one step further to obtain the answer. Instead, the student adopted a new strategy of using congruent triangles to establish the equality of the sides.

Group C

Student: *" A,... I'll call the centre O, AO is equal to OC and DO is equal to OB therefore EO equals OG and HO equals OF and angle EOH equals angle FOG and therefore triangle EOH is equal to triangle FOG therefore EH equals FG."*

Students in this category chose to use congruent triangles and gave no indication of alternative strategies. This student is confident with congruency and applied the idea efficiently and accurately. The responses were very much within the context of the question. There was no indication of an underlying general principle.

Group D

Student: *"Call the centre O. Angle DOA equals angle COB because they are vertically opposite. Oh no! I don't know any more. I was going to say they're similar but you can't really say that. Angle...I suppose angle D equals angle B, Oh! hang on, angle COD, no, angle ADO equals angle CBO, and angle DAO equals BCO, so those triangles are similar sides are equal so they are congruent, apart from that I don't know. I suppose those extended bits are equal so they are equal so they are congruent so, angle, I mean triangle HOE is congruent to FOG, so sides are equal.'"*

Responses in this category did not have a clear overview of what was required to complete the question. This often resulted in students losing track of the solution process or, as in the example provided, a sequential series of steps was given in which the student was not clear of the endpoint until it was reached. There seemed to be little trouble with the congruency concept. The main issue for students who responded this way was keeping in mind all relevant elements of the question and solution process.

Group E

Student: (Student marks O for the intersection of the diagonals and indicates on the diagram that the diagonals bisect each other) *"BOC and AOD will be same because they will be similar."*

Interviewer: *"Similar triangles you are saying?"*

Student: *"Yeah, I think - Oh no! They could be, they'll be congruent as well."*

Interviewer: *"What's making them congruent?"*

Student: *"Um ...the alternate angles* (student means vertically opposite), *is that right? Those give us side angle side."*

Interviewer: *"You've already got three sides anyway".*

Student: *"Oh! I do too! Which means that ABO and DOC will be the same and..."*

Interviewer: *"We're wanting to prove this equals this."*

Student: *"Yeah..umm ..I know FO and HO are equal but and also GO and EO but I don't know how it's going to help me."*

Interviewer: *"Fine. Anything else you know for sure? What do you need? What are you going to do with them?"*

Student: *"I don't know."*

Similarly, the responses given in this group demonstrated an ability with the concept of congruence. However, the application of congruence in a non-prompted situation placed them under pressure. This resulted in the students losing sight of the purpose of the question. In the case recorded above the student was unable to continue even though she was close to having all the elements she needed.

Group F

Student: *"All angles in ABCD are equal, well opposite angles are equal anyway, well the corner E and G are equal distances from the parallelogram from A and C and B and F are equal and BF and DH are same distance so it's just like expanding moving parallelogram out say AE equals 2 cm, just like moving parallelogram opposite corners makes sides opposite each other same distance, same length."*

The response above was the final part of a longer discussion in which the student spent much of the time making assumptions about the parallelogram he was meant to be establishing. The students in this group were convinced that the result "has to be true" or, as in one case, because "CG and DH would be different lengths - it (the sides) would put it out of proportion and opposite angles would not be equal then." Overall these responses are characterised by a form of natural logic although no attempt at deductive reasoning is made.

Group G

Student: *"AD is going to equal BC because it's a parallelogram. BA equals CD, um, no I can't work out how to do it."*

Interviewer: *"What do you know about the diagonals of a parallelogram?"*

Student: *"They are equal."*

(A short discussion followed)

Interviewer: *"What happens when they meet?"*

Student: *"They cause a right angle."*

This group of responses focuses on the properties (often incorrectly) of the given parallelogram. There is no attempt to address the question and the answers are usually left as a series of statements about equal angles, equal sides and parallel sides. Often prompting by the interviewer, as shown above, creates further problems.

Group H

Student: *"I know that angle H to where the angles meet."*

Interviewer: *"Call it O or X."*

Student: *"Vertically opposite ones are equal..(pause)"*

Interviewer: *"Is that going to help you?"*

Student: *"I'm not using the parallelogram - I don't think. No , I don't know."*

This is a typical response in this group where the student cannot make any real attempt to solve the problem. This particular student has seen the angles in the centre of the diagram but this is stated as an observation only.

One further feature of the responses to the question deserves comment . This occurred when eight students, whose overall answer suggested deductive thinking, could not spontaneously use the property: diagonals of a parallelogram bisect each other. Before summarising the general findings of the study in the light of the Solo Taxonomy a brief consideration of the strategies employed by this group of students (referred to as Group I) is discussed.

Group I

Two students first established that the triangles ADO and CBO were congruent. One of the students then established that the diagonals bisect each other and went on to complete the proof. This response was equivalent to those in Group C. The other student was unable to draw a conclusion from the congruent triangles and gave up. This response was similar to those in Group E. Three students, after some limited exploration asked the interviewer if the diagonals of a parallelogram bisect each other. When this was confirmed they completed the question as those in Group C.

Of the remaining three students, one student chose to focus on the two larger triangles in each figure. He knew that triangles ABC and ADC were congruent and was attempting to prove that the corresponding triangles in the larger figure were also congruent. He could not complete the question using this strategy and gave up. The other two students focussed on the quadrilaterals AEHD and CGFB. They attempted to prove these figures congruent by using the incorrect notion that if three sides of a quadrilateral are equal then the fourth side must also be equal.

Summary

The results of the study are summarised in Table 1. There is a clear dichotomy between groups A - E and groups F - H. The former have attempted to provide some deductive argument while the latter groups have not. In terms of the van Hiele Theory students in groups A - E would be considered to be at Level 4, while groups F - H would be at Level 3 and Level 2, respectively.

Group	Description of the response	SOLO	van Hiele Level
A	Concise use of sufficiency aspect: Diagonals bisect therefore parallelogram, therefore EH = FG	Unistructural (second cycle) Formal Mode	4
B	Not as succinct as A above and required some clarification or after an initial attempt at an A response reverted to C below	Transition Formal Mode	4
C	Used congruent triangles, confident and concise response	Relational Formal Mode	4
D	Lacks clear overview of the question. Often unable to keep track of all relevant features of the question and solution process	Multistructural Formal Mode	4
E	Attempted to use congruency without consideration of why it was being employed	Unistructural Formal Mode	4
F	Visual response, use of natural logic, fixed on the look of the diagram and makes no attempt at deductive reasoning	Relational Concrete Symbolic	3
G	Focus on properties of parallelogram and made no attempt to solve the problem	Multistructural Concrete Symbolic	2
H	Focus on one aspect or property	Unistructural Concrete Symbolic	2

Table 1: Summary of responses

Conclusion

This study has provided new evidence of the quality or depth of thinking that indicates the early stages of van Hiele's Level 4. This work highlights the futility of seeing Level 4 thinking as an either/or situation. With the help of the SOLO Taxonomy a pattern of growth and a means of distinguishing between the quality of responses at this Level was identified.

The coding of responses within the SOLO Taxonomy has also brought to the fore the possibility of more than one cycle within the Formal mode (or Level 4). This finding supports other work (see Pegg (1992b)) in which more than one unistructural - multistructural - relational cycle has been identified within a mode. This finding opens the way to a greater understanding of the nature and complexity of Formal (or Level 4) responses by providing the challenge to develop questions that can be responded to at a multistructural or relational level in the second cycle.

The study highlights further the potentially undesirable consequences of focussing on student thinking rather than on student responses. This was made clear when eight students did not spontaneously use the property: the diagonals in a parallelogram bisect each other. This single piece of information was seen to affect the quality of the students' responses and if left unprompted would have resulted in a low van Hiele Level being assigned.

Finally, the results of this study, imply that the application of necessary and sufficient conditions (in the form of definitions) is more difficult to master than competence with congruency concepts. Given the number of definitions that can be used for a parallelogram and the differences in familiarity that students have with these different definitions this finding needs careful probing. Another important aspect that could have a bearing on this work is how the learning context, i.e., how the syllabus is structured and taught, influences such a finding. Questions addressing these issues are currently under analysis.

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